Expansion of Spherical Bessel Functions in a Series of Chebyshev Polynomials

By A. M. Arthurs and R. McCarroll

1. Introduction. In many problems of physics it is necessary to evaluate the spherical Bessel functions over a wide range of argument and up to a high order. For example, their evaluation is necessary in the solution of the integral equations of atomic scattering, as described in the work of Frazer [1].

One method is to generate the functions by means of recurrence formulas [2], [3], [4] which basically provide a large number of orders at a single value of the argument. A method which in essence provides a single order for a given *range* of argument, and which is suitable for use in automatic computations associated with atomic scattering, is to expand the spherical Bessel functions in terms of Chebyshev polynomials

2. The Method. We introduce the shifted Chebyshev polynomial $T_n^*(z)$ which satisfies the following differential equation given by Lanczos [5].

$$(1)\qquad (z - z^2)\frac{d^2T_n^*}{dz^2} - \frac{(2z-1)}{2}\frac{dT_n^*}{dz} + n^2T_n^* = 0 \qquad 0 \leqq z \leqq 1.$$

The spherical Bessel function

$$(2)\qquad j_r(x) = (\pi/2x)^{1/2} J_{r+1/2}(x)$$

is expanded in series of $T_n^*(z)$ for $z \leqq 1$ as

$$(3)\qquad j_r(x) = (x/2)^r N_r \sum_n A_n T_n^*(z)$$

where

$$(4)\qquad z = x^2/p$$

and where N_r is the normalization factor given by

$$(5)\qquad N_r = \pi^{1/2}[2\,\Gamma(r + 3/2)\sum_n (-1)^n A_n]^{-1},$$

chosen so that as $x \to 0$

$$(6)\qquad j_r(x) \to \pi^{1/2}(x/2)^r/2\,\Gamma(r + 3/2).$$

The parameter p is chosen according to the required x-range.

Substituting the expression (3) into the differential equation satisfied by $j_r(x)$, namely,

$$(7)\qquad \left[\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + 1 - \frac{r(r+1)}{x^2}\right]j_r(x) = 0,$$

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TABLE 1

Tables of Expansion Coefficients

$j_r(x) = (x/2)^r N_r \sum_n A_n T_n^*(x^2/100)$ $-10 \leq x \leq 10$

n	j_0 A_n	j_1 A_n
0	1.0000 0000 0000	1.0000 0000 0000
1	−1.8370 2985 7106	−1.7335 5787 9572
2	1.6181 7789 5224	1.7037 9637 8855
3	−2.4956 2512 4954	−1.2117 2623 8820
4	1.6832 2166 9128	0.4966 0881 5456
5	−0.5891 1635 2032	−0.1261 1219 3614
6	0.1275 7849 3735	0.0216 6172 6076
7	−0.0189 6827 0697	−0.0026 8726 9369
8	0.0020 6883 1574	0.0002 5255 1305
9	−0.0001 7326 9014	−0.0000 1863 3794
10	0.0000 1152 2733	0.0000 0110 9455
11	−0.0000 0062 4256	−0.0000 0005 4481
12	0.0000 0002 8116	0.0000 0000 2246
13	−0.0000 0000 1070	−0.0000 0000 0079
14	0.0000 0000 0035	0.0000 0000 0002
15	−0.0000 0000 0001	

n	j_2 A_n	j_3 A_n
0	1.0000 0000 0000	1.0000 0000 0000
1	−1.8364 1601 5317	−1.7972 4765 8075
2	1.4239 0223 9665	1.1197 5560 6070
3	−0.7040 0124 0486	−0.4322 1712 7394
4	0.2158 5586 1952	0.1074 1449 1493
5	−0.0438 6700 5066	−0.0183 5466 7482
6	0.0063 0718 4207	0.0022 8121 8406
7	−0.0006 7512 0822	−0.0002 1543 2710
8	0.0000 5593 1115	0.0000 1599 1357
9	−0.0000 0369 5921	−0.0000 0095 8213
10	0.0000 0019 9498	0.0000 0004 7352
11	−0.0000 0000 8968	−0.0000 0000 1964
12	0.0000 0000 0341	0.0000 0000 0069
13	−0.0000 0000 0011	−0.0000 0000 0002

n	j_4 A_n	j_5 A_n
0	1.0000 0000 0000	1.0000 0000 0000
1	−1.6836 0574 1497	−1.5488 1806 4538
2	0.8722 0517 6005	0.6859 0242 4243
3	−0.2782 8514 3648	−0.1872 6092 6662
4	0.0585 4062 0216	0.0342 7629 1788
5	−0.0086 6953 0055	−0.0044 9108 8777
6	0.0009 5173 1760	0.0004 4233 6046
7	−0.0000 8057 5832	−0.0000 3398 2101
8	0.0000 0542 5189	0.0000 0209 5486
9	−0.0000 0029 7662	−0.0000 0010 6104
10	0.0000 0001 3573	0.0000 0000 4494
11	−0.0000 0000 0523	−0.0000 0000 0162
12	0.0000 0000 0017	0.0000 0000 0005

n	j_6 A_n	j_7 A_n
0	1.0000 0000 0000	1.0000 0000 0000
1	−1.4171 3242 2748	−1.2972 4767 3744
2	0.5479 5705 4665	0.4451 8041 2800
3	−0.1310 4234 0105	−0.0948 5564 9030
4	0.0212 7626 4218	0.0138 5756 5596
5	−0.0025 0394 3447	−0.0014 8187 0216
6	0.0002 2390 8476	0.0001 2144 5196
7	−0.0000 1575 8616	−0.0000 0789 1023
8	0.0000 0089 6941	0.0000 0041 7246
9	−0.0000 0004 2186	−0.0000 0001 8329
10	0.0000 0000 1669	0.0000 0000 0680
11	−0.0000 0000 0056	−0.0000 0000 0022
12	0.0000 0000 0002	0.0000 0000 0001

Tables of Expansion Coefficients—Continued

n	$A_n^{j_8}$	$A_n^{j_9}$
0	1.0000 0000 0000	1.0000 0000 0000
1	−1.1911 3494 9922	−1.0981 6702 5095
2	0.3675 0203 3005	0.3078 0618 7533
3	−0.0706 7640 7499	−0.0539 7859 5631
4	0.0093 9388 4497	0.0065 8545 8180
5	−0.0009 2119 2296	−0.0005 9665 6926
6	0.0000 6972 0100	0.0000 4196 8280
7	−0.0000 0420 9178	−0.0000 0236 6972
8	0.0000 0020 7889	0.0000 0010 9704
9	−0.0000 0000 8569	−0.0000 0000 4260
10	0.0000 0000 0300	0.0000 0000 0141
11	−0.0000 0000 0009	−0.0000 0000 0004

n	$A_n^{j_{10}}$	$A_n^{j_{11}}$
0	1.0000 0000 0000	1.0000 0000 0000
1	−1.0168 7201 6138	−0.9456 3737 0019
2	0.2611 5956 8765	0.2241 3285 9656
3	−0.0421 0983 2180	−0.0334 5827 7279
4	0.0047 5003 6264	0.0035 1079 6899
5	−0.0004 0010 7956	−0.0002 7638 9359
6	0.0000 2629 5870	0.0000 1705 0581
7	−0.0000 0139 1880	−0.0000 0085 0428
8	0.0000 0006 0782	0.0000 0003 5115
9	−0.0000 0000 2232	−0.0000 0000 1223
10	0.0000 0000 0070	0.0000 0000 0036
11	−0.0000 0000 0002	−0.0000 0000 0001

n	$A_n^{j_{12}}$	$A_n^{j_{13}}$
0	1.0000 0000 0000	1.0000 0000 0000
1	−0.8829 7010 1098	−0.8275 7427 4817
2	0.1943 1493 5452	0.1699 8535 0546
3	−0.0270 1121 0330	−0.0221 1284 8562
4	0.0026 5020 1752	0.0020 3772 5111
5	−0.0001 9588 4596	−0.0001 4196 6316
6	0.0000 1138 8223	0.0000 0780 5415
7	−0.0000 0053 7112	−0.0000 0034 9193
8	0.0000 0002 1036	0.0000 0001 3008
9	−0.0000 0000 0697	−0.0000 0000 0411
10	0.0000 0000 0020	0.0000 0000 0011

n	$A_n^{j_{14}}$	$A_n^{j_{15}}$
0	1.0000 0000 0000	1.0000 0000 0000
1	−0.7783 5629 2813	−0.7344 0438 9543
2	0.1498 9703 5766	0.1331 3186 9771
3	−0.0183 2686 5868	−0.0153 5579 5806
4	0.0015 9235 3331	0.0012 6227 9463
5	−0.0001 0493 0348	−0.0000 7891 7018
6	0.0000 0547 2793	0.0000 0391 5367
7	−0.0000 0023 2885	−0.0000 0015 8871
8	0.0000 0000 8272	0.0000 0000 5393
9	−0.0000 0000 0250	−0.0000 0000 0156
10	0.0000 0000 0006	0.0000 0000 0004

TABLE 2
Table of Normalization Factors

r	N_r
0	$10^{-1}\cdot 1.0670\ 1130\ 3957$
1	$10^{-1}\cdot 1.0588\ 0224\ 7273$
2	$10^{-2}\cdot 5.0977\ 3196\ 1602$
3	$10^{-2}\cdot 1.7016\ 2862\ 5983$
4	$10^{-3}\cdot 4.3387\ 2970\ 2098$
5	$10^{-4}\cdot 8.8939\ 6388\ 4585$
6	$10^{-4}\cdot 1.5178\ 7598\ 0079$
7	$10^{-5}\cdot 2.2135\ 3657\ 5215$
8	$10^{-6}\cdot 2.8143\ 4241\ 8551$
9	$10^{-7}\cdot 3.1696\ 2427\ 4560$
10	$10^{-8}\cdot 3.2028\ 4879\ 3012$
11	$10^{-9}\cdot 2.9343\ 5222\ 5384$
12	$10^{-10}\cdot 2.4587\ 5268\ 1163$
13	$10^{-11}\cdot 1.8981\ 3134\ 3689$
14	$10^{-12}\cdot 1.3584\ 9275\ 4731$
15	$10^{-14}\cdot 9.0623\ 7664\ 9006$

and using the properties of the shifted Chebyshev polynomials, we obtain a set of simultaneous linear algebraic equations which may be solved for the ratios A_1/A_0, A_2/A_0, $\cdots$. The solutions may be normalized, using the factors given by (5).

3. Calculations and Results. The calculation of the coefficients A_n has been carried out on a DEUCE digital computer for $p = 100$, and $r = 0$ to 15. For all values of r, A_0 has been chosen as 1.0, and the ratios A_n/A_0 computed accordingly. Tables of the coefficients, together with the corresponding normalization factors are presented herein. The expansion coefficients are given to 12 decimal places to insure that for the range of x considered the spherical Bessel functions should be accurate to 10 significant figures.

As can be seen from the tables, the convergence of the coefficients is very rapid; if the Chebyshev expansion and Taylor series are curtailed after n terms, the ratio of the $(n + 1)$th terms is about $1/2^{n-1}$.

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The Queen's University of Belfast
N. Ireland

Mathematical Institute
Oxford, England

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